

VIRTUAL KNOT GROUPS

SE-GOO KIM

ABSTRACT. Virtual knots, defined by Kauffman, provide a natural generalization of classical knots. Most invariants of knots extend in a natural way to give invariants of virtual knots. In this paper we study the fundamental groups of virtual knots and observe several new and unexpected phenomena.

In the classical setting, if the longitude of a knot is trivial in the knot group then the group is infinite cyclic. We show that for any classical knot group there is a virtual knot with that group and trivial longitude. It is well known that the second homology of a classical knot group is trivial. We provide counterexamples of this for virtual knots.

For an arbitrary group G , we give necessary and sufficient conditions for the existence of a virtual knot group that maps onto G with specified behavior on the peripheral subgroup. These conditions simplify those that arise in the classical setting.

1. INTRODUCTION

In 1996 Kauffman presented the theory of Virtual Knots. In [9] he described this very natural generalization of classical knot theory and began to develop the theory of virtual knotting. A significant recent accomplishment is the work of Goussarov, Polyack, and Viro [7] in which it is shown that the entire theory of finite type invariants of knots extends naturally to the realm of virtual knots. In this paper we explore the fundamental groups of virtual knots, focusing on properties of the peripheral subgroup and on the homology theory of these groups.

The core idea of virtual knot theory is easily described. To each knot diagram in \mathbb{R}^3 one can form an associated Gauss diagram, a simple diagram that captures all the crossing information of the knot diagram. Details will be given later. Knot diagrams determine isotopic knots if they are related by Reidemeister moves; there are corresponding moves on Gauss diagrams that generate an equivalence relation on the set of Gauss diagrams. It can be shown that if two knots determine equivalent Gauss diagrams, they represent the same knot.

A virtual knot is defined to be an equivalence class of Gauss diagrams. It follows from the discussion above that every knot determines a unique virtual knot and if two knots determine the same virtual knot then they are in fact the same knot. However, not every virtual knot arises from a knot and hence virtual knot theory offers a nontrivial extension of classical knot theory. The work of [9, 7] demonstrates that much of classical knot theory extends to the virtual setting. We will see here that while the theory of knot groups does extend in a natural way as well, a number of new phenomena arises that contrasts sharply with what occurs in the classical setting.

To each virtual knot K there is associated a fundamental group Π_K along with a peripheral subgroup, generated by a meridian and longitude, m and l . We observe here the relatively simple fact that as in the classical case, l and m commute.

1991 *Mathematics Subject Classification.* Primary 57M25.

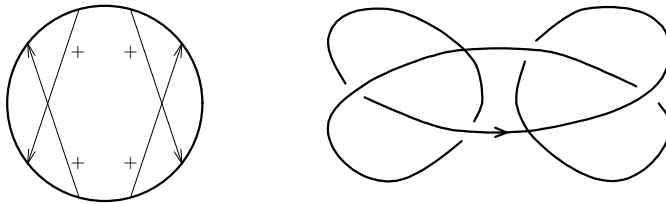


FIGURE 1. A Gauss diagram and its corresponding virtual knot

Classical knot groups have Wirtinger presentation of deficiency 1 and all the group homology of dimension greater than 1 is trivial. However, we observe that virtual knot groups may have Wirtinger presentations of deficiency 0 and that any Wirtinger group of deficiency 0 or 1 can be realized as a virtual knot group. From this, we can find examples of virtual knot groups with nontrivial second homology.

It is well known that a classical knot with trivial longitude is the unknot. We observe that any Wirtinger group of deficiency 1 is the group of some virtual knot with trivial longitude. As a corollary, any classical knot group is the group of a virtual knot with trivial longitude, which gives examples of nontrivial virtual knots with trivial longitude.

Let G be a group and let μ and λ be elements of G . Is there a virtual knot K and a surjective homomorphism $\rho: \Pi_K \rightarrow G$ such that $\rho(m) = \mu$ and $\rho(l) = \lambda$, where m and l are the meridian and longitude of K ? In the classical knot case, Edmonds and Livingston [3] answered this for $G = S_n$, and Johnson and Livingston [8] extended this result to general group representations. In the virtual case, only one of their conditions is needed, namely, that the image of longitude commutes with a normal generator of G . In the course of proving this, we examine properties of connected sums and present an example of a nontrivial virtual knot which is a connected sum of two trivial virtual knots.

If g_1 and g_2 are commuting elements of a group Π , then their Pontryagin product, defined later, is an element $\langle g_1, g_2 \rangle$ in $H_2(\Pi)$. We observe that the second homology group of a virtual knot group is generated by the Pontryagin product of the meridian and longitude of the virtual knot. In the classical case, the Pontryagin product is zero since the second homology groups of classical knots are zero. We finally present examples of virtual knots whose groups have the second homology \mathbb{Z} or $\mathbb{Z}/2$, and an example of a virtual knot which has trivial longitude and whose group is of deficiency 0.

ACKNOWLEDGEMENTS. The author would like to thank Chuck Livingston for his suggestions and discussions.

2. VIRTUAL KNOT GROUPS

2.1. Gauss diagrams. Knots are usually presented by knot diagrams, that is, generic immersions of the circle into the plane enhanced by information on overpasses and underpasses at double points. A generic immersion of a circle into the plane is characterized by its *Gauss diagram*, which consists of the circle together with the preimages of each double point of the immersion connected by a *chord*. To incorporate the information on overpasses and underpasses, the chords are oriented from the upper branch to the lower one. Furthermore, each chord is equipped with

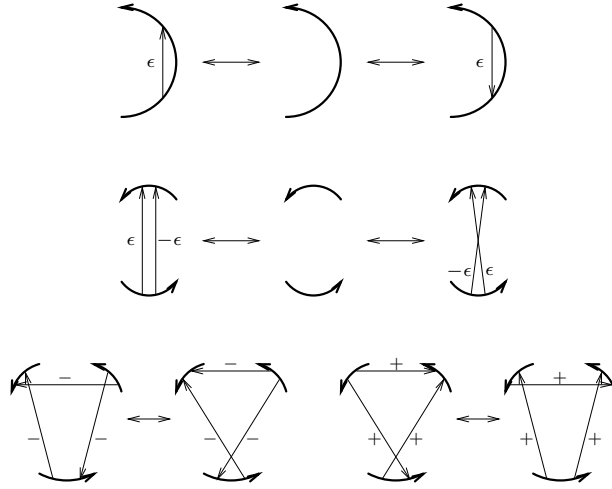


FIGURE 2. Moves of Gauss diagrams corresponding to Reidemeister moves.

the sign of the corresponding double point (local writhe number). The result is called a *Gauss diagram* of the knot. Thus Gauss diagrams can be considered as an alternative way to present knots. However, not every Gauss diagram is indeed a Gauss diagram of some knot. For example, see Figure 1 and [9, 7].

2.2. Virtual knots. As is well-known, when a knot changes by a generic isotopy, its diagram undergoes a sequence of Reidemeister moves. Figure 2 depicts the counter-parts of the Reidemeister moves for Gauss diagrams. All moves corresponding to the first and second Reidemeister moves are shown in the top and middle rows, respectively. Though there are eight moves corresponding to the third Reidemeister moves, it is known that only two moves of them are necessary; they are shown in the bottom row in Figure 2. For details, see [7]. A sequence of moves in Figure 2 is called an *isotopy* for Gauss diagrams. A *virtual knot* is defined as an equivalence class of Gauss diagrams up to isotopy. Goussarov, Polyak, and Viro [7] proved that if two classical knots determine the same virtual knot, they are isotopic in the classical sense.

2.3. Virtual knot groups. Kauffman [9] has proved that many isotopy invariants of knots extend naturally to invariants of virtual knots. In particular, the notion of knot group extends in a straightforward manner, disregarding the original topological nature. The knot group, which is defined for classical knots as the fundamental group of the knot complement, is extended via a formal construction of a Wirtinger presentation (defined later). This construction can be written down in terms of a Gauss diagram as follows.

Let D be a Gauss diagram. If we cut the circle at each arrowhead (forgetting arrowtails), the circle of D is divided into a set of arcs. To each of these arcs there corresponds a generator of the group. Each arrow gives rise to a relation. Suppose the sign of an arrow is ϵ , its tail lies on an arc labeled a , its head is the final point of an arc labeled b and the initial point of an arc labeled c . Then we assign to this arrow the relation $c = a^{-\epsilon}ba^{\epsilon}$. (For simplicity, u^v will denote $v^{-1}uv$ for any words u

and v .) The resulting group is called the *group of the Gauss diagram*, denoted Π_D . One can check that it is invariant under the Reidemeister moves of Gauss diagrams shown in Figure 2 and that the group of a Gauss diagram obtained from a classical knot is the fundamental group of that knot. Therefore, the notion of knot group is extended to virtual knots. The *group of a virtual knot* is defined as the group of a representative Gauss diagram of the virtual knot. For details, see [9, 7].

2.4. Peripheral subgroups. The notion of peripheral subgroup system also extends. For the meridian, take the generator corresponding to any of the arcs. To write down the longitude, we go along the circle starting from this arc and write a^ϵ , when passing the head of an arrow whose sign is ϵ and whose tail lies on the arc labeled a , and finally write t^{-p} , where t is the meridian of the starting arc and p is the number for which the result is to be in the commutator subgroup of the virtual knot group. The choice of such a number p is possible since the abelianization of a virtual knot group is cyclic. One can easily check that meridian and longitude are uniquely determined up to conjugation under the Reidemeister moves. The *peripheral subgroup* of a virtual knot is the subgroup generated by its meridian and longitude.

2.5. Wirtinger presentations. Consider a group presentation of the form

$$\langle t_1, \dots, t_p \mid r_1, \dots, r_q \rangle,$$

where each relator r_k , $k = 1, \dots, q$, is of the form $t_i^{-1}t_j^{w_k}$, for some i and j , $1 \leq i, j \leq p$, and some word w_k in t_1, \dots, t_p . If all t_i are conjugate, such a group presentation is called a *Wirtinger presentation with respect to t* , where t is any element in the conjugacy class of the t_i . Such an element t is said to *normally generate the group*, or the group is said to be *of weight 1*. A group is called *Wirtinger* if it has a Wirtinger presentation. Since the first homology group of a group is the abelianization of the group, the first homology group of a Wirtinger group is infinite cyclic \mathbb{Z} .

2.6. Deficiency of a group presentation. For a Gauss diagram D , it is clear by definition that its group Π_D has a Wirtinger presentation with the number of generators equal to the number of relators. The *deficiency* of a group presentation is the number of generators minus the number of relators. The group Π_D (hence any virtual knot group) then has a Wirtinger presentation of nonnegative deficiency.

Proposition 1. *Any virtual knot group has a Wirtinger presentation of deficiency 0 or 1 and hence its second homology group is cyclic.*

Proof. Let Π have a Wirtinger presentation of nonnegative deficiency d . Then the group Π has a presentation with $n + d$ generators t_i and n relators r_j . Define a 2-dimensional CW complex X_Π^2 as follows: the 1-skeleton of X_Π^2 is a one-point union of $n + d$ circles, each of which represents each generator t_i , and n 2-disks are attached to the 1-skeleton along the relators r_j . By the Van Kampen theorem, Π is the fundamental group of X_Π^2 . Thus, $H_1(X_\Pi^2) = \mathbb{Z}$ and $\text{rank}(H_1(X_\Pi^2)) = 1$. The cellular chain complex of X_Π^2 is $\dots \rightarrow 0 \rightarrow \mathbb{Z}^n \xrightarrow{\partial_2} \mathbb{Z}^{n+d} \xrightarrow{\partial_1} \mathbb{Z}$, where ∂_1 is a zero map, and thus $d \leq \text{rank}(\text{coker } \partial_2) = \text{rank}(H_1(X_\Pi^2)) = 1$. In conclusion, the group Π_D of a Gauss diagram D (hence any virtual knot group) has a Wirtinger presentation of deficiency 0 or 1.

By rank arguments similar to the one above, $H_2(X_\Pi^2)$ is zero and infinite cyclic if the deficiency is 1 and 0, respectively. Continuing attaching higher cells to X_Π^2 we get an Eilenberg-MacLane space $K(\Pi, 1)$ whose 2-skeleton is X_Π^2 . Thus, the group homology $H_2(\Pi)$, which is defined as $H_2(K(\Pi, 1))$, is cyclic since it is a quotient of $H_2(X_\Pi^2)$. \square

3. REALIZATION

In this section, we show that any Wirtinger group Π of deficiency 0 or 1 can be realized as a virtual knot group. In the classical case, if $H_2(\Pi) = 0$, Kervaire [10] showed that $\Pi \cong \pi_1(S^n - S^{n-2})$ for some smooth $(n-2)$ -sphere S^{n-2} in S^n , $n \geq 5$. However, Kervaire's theorem does not apply to the case $n = 3$ and there are Wirtinger groups of deficiency 1 which are not classical knot groups. Also, if $H_2(\Pi) \neq 0$, then Π can never be realized as a classical knot group. For details, see [5, 10].

3.1. Cyclic and realizable Wirtinger presentations. A *cyclic* Wirtinger presentation is a Wirtinger presentation of the form $\langle t_1, \dots, t_n \mid r_1, \dots, r_n \rangle$, where the j -th relator r_j is of the form $t_{j+1}^{-1} t_j^{w_j}$ for $j = 1, \dots, n \pmod{n}$, and some word w_j in t_1, \dots, t_n . A cyclic Wirtinger presentation can be transformed into a special kind of cyclic Wirtinger presentation, called a *realizable* Wirtinger presentation, with the property that each w_j is a one-letter word t_k^ϵ for some $k = 1, \dots, n$ and $\epsilon = \pm 1$, by introducing more generators and relators. For example, if $w_1 = t_3 t_7^{-1} t_5$ and $t_2 = t_1^{w_1}$, then we introduce two more generators, say, t'_1, t'_2 , remove the relator r_1 and add three more relators $t_1'^{-1} t_1^{t_3}$, $t_2'^{-1} (t_1')^{t_7^{-1}}$, and $t_2'^{-1} (t_2')^{t_5}$. It is easy to see that the new group presentation presents the same group. Note that both cyclic and realizable Wirtinger presentations have deficiency 0.

Lemma 2. *A Wirtinger presentation of deficiency 0 or 1 can be transformed to a realizable Wirtinger presentation.*

Proof. Let Π be a Wirtinger presentation of deficiency 0 or 1, say, $\Pi = \langle t_1, \dots, t_n \mid r_1, \dots, r_m \rangle$, where $m = n$ or $n - 1$. If $m = n - 1$, by doubling the relator r_m , we may assume $m = n$. By the arguments prior to this lemma, it suffices to construct a cyclic Wirtinger presentation from Π .

Let P_Π be a graph with n vertices $\{v_1, \dots, v_n\}$ and n edges corresponding to relators in the way that an edge has end vertices v_i and v_j if and only if there is a relator of the form $t_i^{-1} t_j^w$. Such an edge is denoted by e_j^i . Since all t_i are conjugate, the graph P_Π is connected. If two edges e_j^i and e_k^j meet at a vertex v_j , then we have two relators $t_i^{-1} t_j^{w_1}$ and $t_j^{-1} t_k^{w_2}$, or $t_i = t_j^{w_1}$ and $t_j = t_k^{w_2}$. This implies $t_i = w_1^{-1} w_2^{-1} t_k w_2 w_1 = (w_2 w_1)^{-1} t_k w_2 w_1$. We now remove the relator $t_i^{-1} t_j^{w_1}$ and add $t_i^{-1} t_k^{w_2 w_1}$ to get a new presentation. It is obvious that the new presentation presents the same group. This operation corresponds to an operation on the graph P_Π of deleting an edge e_j^i and adding an edge e_k^i .

Recall that a cycle of a graph is a simply closed path on the graph. Using the above operation on the graph P_Π we will construct a cycle from P_Π . Since P_Π has betti number 1, it has one and only one cycle C . We will use induction on the length l of C . If $l = n$, then $C = P_\Pi$ and thus P_Π is a cycle. Suppose $l < n$. Then there is a vertex v_i which is not on C . Since P_Π is connected, there is a path from v_i to a vertex of C . On this path, there is an edge e_k^j such that v_j is not in C but

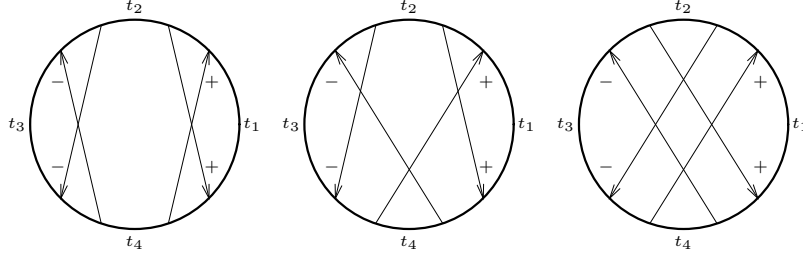


FIGURE 3. Gauss diagrams constructed from a group presentation
 $\langle t_1, t_2, t_3, t_4 \mid t_2^{-1}t_1^{t_4}, t_3^{-1}t_2^{t_4^{-1}}, t_4^{-1}t_3^{t_2^{-1}}, t_1^{-1}t_4^{t_2} \rangle$

v_k is in C . By the operation described above, we can construct P'_Π with a cycle C' containing all vertices of C and v_j . Since the length of C' is greater than the length of C , we keep doing this process to get a cycle.

The corresponding operations on the Wirtinger presentation of Π give a cyclic Wirtinger presentation. This completes the proof. \square

Theorem 3. *Any Wirtinger presentation of deficiency 0 or 1 can be realized as a virtual knot group.*

Proof. By Lemma 2, we can assume that a realizable Wirtinger presentation with n generators t_i and n relators r_i is given. We start with a circle. Choose n points on the circle, which divide the circle into n arcs labeled t_1, \dots, t_n , successively counterclockwise. For each $i = 1, \dots, n$, if $r_i = t_{i+1}^{-1}t_i^{\epsilon}$, attach an oriented chord with sign ϵ from a point on the arc labeled t_j to the point dividing t_i and t_{i+1} . By definition, the group of this Gauss diagram has the given presentation. \square

Example 4. The Gauss diagrams in Figure 3 are constructed from a Wirtinger presentation $\langle t_1, t_2, t_3, t_4 \mid t_2^{-1}t_1^{t_4}, t_3^{-1}t_2^{t_4^{-1}}, t_4^{-1}t_3^{t_2^{-1}}, t_1^{-1}t_4^{t_2} \rangle$. This presentation can be transformed to $\langle t_2, t_4 \mid t_2t_4t_2 = t_4t_2t_4 \rangle$ that is a presentation of the trefoil group. Note also that the longitude is trivial. The first and second diagrams have bracket polynomial $-A^{-6} - A^{-4} + A^{-2} + 3 + A^2 - A^4 - A^6$ while the third has unit bracket polynomial 1. Thus, there are virtual knots with the same peripheral structure but different bracket polynomials. The third is also an example of nontrivial virtual knot with unit bracket polynomial. For the notion of the bracket polynomial, see [9].

3.2. On peripheral subgroups. Let D be a Gauss diagram with group $\Pi_D = \langle t_1, \dots, t_n \mid r_1, \dots, r_n \rangle$, where $r_i = t_{i+1}^{-1}t_i^{w_i}$ for each $i = 1, \dots, n \pmod{n}$ and some word w_i in t_1, \dots, t_n . Then the longitude of D is $l = w_1 \cdots w_n t_1^{-p}$, where p is chosen so that l is in the commutator subgroup of Π_D .

Proposition 5. *The longitude of D commutes with the meridian t_1 and hence the peripheral subgroup is abelian.*

Proof. Iterating all relators r_1, \dots, r_n , we have $t_1 = w_n^{-1} \cdots w_1^{-1} t_1 w_1 \cdots w_n$, or t_1 commutes with the longitude. This then shows that the peripheral subgroup of Π_D is an abelian group generated by the meridian and longitude. \square

Proposition 6. *If a Wirtinger presentation has deficiency 1 and if λ is a commutator element commuting with a normal generator t , then it is the group of a virtual knot with longitude λ . In particular, a Wirtinger presentation of deficiency 1 can be realized as the group of a virtual knot with trivial longitude.*

Proof. Suppose that a group Π has a Wirtinger presentation of deficiency 1. Then adding a redundant relator $r_n = t_1^{-1}w_1 \cdots w_{n-1}t_nw_{n-1}^{-1} \cdots w_1^{-1}$ to the presentation gives a Wirtinger presentation of deficiency 0 presenting the same group. We can easily see that the longitude obtained from the new presentation is trivial. Moreover, in this case, if λ is a commutator element commuting with t_1 in Π , adding $r_n = t_1^{-1}\lambda^{-1}w_1 \cdots w_{n-1}t_nw_{n-1}^{-1} \cdots w_1^{-1}\lambda$ to the original presentation, we have longitude λ for the virtual knot obtained from the presentation. \square

Corollary 7. *Any classical knot group is the group of a virtual knot with trivial longitude.*

Corollary 8. *There is a nontrivial virtual knot with trivial longitude.*

We will see a partial converse of Proposition 6 in Section 5. An example of a Wirtinger presentation of deficiency 0 having trivial longitude will be presented in Section 6.

4. PERIPHERALLY SPECIFIED HOMOMORPHISMS

Let G be a group and let μ and λ be elements of G . Is there a virtual knot K and a surjective homomorphism $\rho: \Pi_K \rightarrow G$ such that $\rho(m) = \mu$ and $\rho(l) = \lambda$, where m and l are the meridian and longitude of K ? In the classical knot case Edmonds and Livingston [3] answered this for $G = S_n$, and Johnson and Livingston [8] extended this result to general group representations. In this section, we observe that only one of their conditions is needed in the virtual setting, namely, that the image of longitude commutes with a normal generator of G .

4.1. Realizable elements. An initial observation is that G must be finitely generated and of weight one. Fix G and a normal generator μ with these properties. If, for given μ and λ in G , a virtual knot K and representation ρ as above exist, we say λ is *realizable*. Let Λ denote the set of realizable elements. Johnson and Livingston have proved the following in the classical knot case.

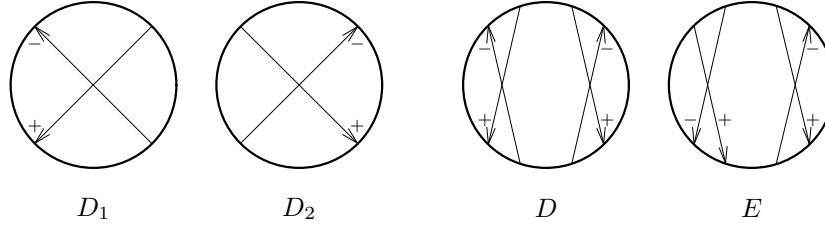
Theorem 9. [8] *The set of realizable by classical knot groups is a nonempty subgroup of G .*

Theorem 10. [8] *λ is realizable by a classical knot if and only if $\lambda \in G'' \cap Z(\mu)$, $\langle \mu, \lambda \rangle = 0 \in H_2(G)$, and $\{\mu, \lambda\} = 0 \in H_3(G/G')/p_*(H_3(G))$.*

In the above statement G'' denotes the second commutator subgroup of G and $Z(\mu)$ is the centralizer of μ . $\langle \cdot, \cdot \rangle$ denotes the Pontryagin product (which will be defined later), which maps pairs of commuting elements of G to $H_2(G)$. $\{ \cdot, \cdot \}$ is a map which sends pairs of elements of G satisfying the first and second conditions to $H_3(G/G')/p_*(H_3(G))$. $p: G \rightarrow G/G'$ is the natural projection. G' denotes the commutator subgroup of G .

The counter-parts of the above two theorems for virtual knots are the following.

Theorem 11. *Λ is a nonempty subgroup of G .*

FIGURE 4. Connected sums of two trivial virtual knots D_1 and D_2

Theorem 12. $\Lambda = G' \cap Z(\mu)$, i.e., a commutator element λ in G is realizable if and only if it commutes with the normal generator μ .

In order to prove these, we introduce a based connected sum of two Gauss diagrams.

4.2. Connected sum. For $i = 1, 2$, let D_i be a Gauss diagram and let p_i be a point of the circle of D_i meeting no chords. We assume that each circle of D_i has a counter-clockwise orientation. The *connected sum of D_1 and D_2 based at (p_1, p_2)* is defined in usual manner: cut small intervals around p_i not intersecting any chords and attach the ends of the intervals according to the orientations.

4.3. Group presentation of a connected sum. Suppose that the groups of two Gauss diagrams D_1 and D_2 are $\Pi_{D_1} = \langle t_1, \dots, t_n \mid t_2^{-1}t_1^{u_1}, \dots, t_1^{-1}t_n^{u_n} \rangle$ and $\Pi_{D_2} = \langle s_1, \dots, s_m \mid s_2^{-1}s_1^{v_1}, \dots, s_1^{-1}s_m^{v_m} \rangle$, respectively, where u_i are words in t_j and v_i are words in s_j . Suppose that p_1 is on the arc represented by t_1 and p_2 is on the arc represented by s_1 . Then the group of the connected sum D of D_1 and D_2 at (p_1, p_2) has a presentation

$$\langle t_1, \dots, t_n, s_1, \dots, s_m \mid t_2^{-1}t_1^{u'_1}, \dots, s_1^{-1}t_n^{u'_n}, s_2^{-1}s_1^{v'_1}, \dots, t_1^{-1}s_m^{v'_m} \rangle,$$

where u'_i is obtained from u_i in such a way that each t_1 in u_i is replaced with s_1 if it is read off from a chord with arrow tail lying on the arc between the end point of the arc t_n and the point p_1 , and v'_i is defined in the similar way.

Example 13. A striking example is a nontrivial virtual knot that is a connected sum of two trivial virtual knots. The Gauss diagram D in Figure 4 has trefoil group while E can be transformed into a trivial Gauss diagram by Reidemeister moves. This also says that an unbased connected sum of two virtual knots is not uniquely defined.

4.4. Representation of a connected sum. Let $\rho_i : \Pi_{D_i} \rightarrow G$ ($i = 1, 2$) be representations with $\rho_1(t_1) = \rho_2(s_1) = \mu$ and $\rho_i(l_i) = \lambda_i$, where l_i is the longitude to D_i . Define a representation ρ of the group of the connected sum D of D_1 and D_2 at (p_1, p_2) to G by $\rho(t_i) = \rho_1(t_i)$, $i = 1, \dots, n$ and $\rho(s_j) = \rho_2(s_j)$, $j = 1, \dots, m$. Since $\rho_1(t_1) = \rho_2(s_1)$, the representation ρ is well-defined and it satisfies that $\rho(m) = \mu$ and $\rho(l) = \lambda_1\lambda_2$, where $m = t_1$ and l are the meridian and longitude of D . This is the key to proving Theorem 11.

Proof of Theorem 11. If λ_1 and λ_2 are realizable, then a connected sum can be used to show $\lambda_1\lambda_2$ is realizable. To realize λ_1^{-1} , use the Gauss diagram used to realize λ_1 with orientation and all signs of chords reversed. That the set is nonempty follows

from Theorem 9 since a knot group is a virtual knot group. It follows from a method of modifying group representations similar to the ones given in Section 3 as well. \square

Remark 14. Consider a knot K in S^3 with the meridian m and longitude l and a representation $\rho: \pi_1(S^3 \setminus K) \rightarrow G$ such that $\rho(m) = \mu$ and $\rho(l) = \lambda$ for some λ in G . The connected sum of K and its mirror image \bar{K} gives a representation $\rho: \pi_1(S^3 \setminus K \# \bar{K}) \rightarrow G$ mapping the longitude to the identity element in G . In particular, the identity element is realizable by a classical knot group. Note that this follows from Theorem 9 as well.

Proof of Theorem 12. It has been proved in Section 3 that $\Lambda \subset G' \cap Z(\mu)$. To show the converse, let λ be an element in $G' \cap Z(\mu)$. Let K be a knot in S^3 and let $\rho: \pi_1(S^3 \setminus K) \rightarrow G$ be a representation with $\rho(l) = 1$ as seen in the remark prior to this proof. The knot group $\Pi = \pi_1(S^3 \setminus K)$ has a Wirtinger presentation $\langle t_1, \dots, t_n \mid t_2^{-1}t_1^{w_1}, \dots, t_n^{-1}t_{n-1}^{w_{n-1}}, t_1^{-1}t_n^{w_n} \rangle$, where the last relator is redundant, the longitude of K is $l = t_1^{-p}w_1 \cdots w_n$, $\rho(t_1) = \mu$ and $\rho(l) = 1$.

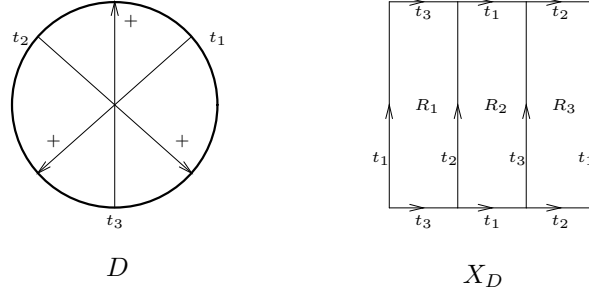
Since ρ is onto and hence the restriction of ρ to Π' is onto G' , there is an element u in Π' with $\rho(u) = \lambda$. Let $\Xi = \langle t_1, \dots, t_n \mid t_2^{-1}t_1^{w_1}, \dots, t_n^{-1}t_{n-1}^{w_{n-1}}, t_1^{-1}t_n^{w_n u} \rangle$. Then Ξ is a quotient group of Π with an extra relator $t_1^{-1}t_n^{w_n u}$. Thus Ξ has a Wirtinger presentation of deficiency 0. Since $lu \in \Xi'$, the group Ξ can be realized as the group of a virtual knot with longitude lu . Define $\bar{\rho}: \Xi \rightarrow G$ by $\bar{\rho}(t_i) = \rho(t_i)$, $i = 1, \dots, n$. Since $\bar{\rho}(t_1^{-1}t_n^{w_n u}) = (\bar{\rho}t_1)^{-1}((\bar{\rho}t_n)^{\bar{\rho}w_n})^{\bar{\rho}u} = \mu^{-1}\mu^\lambda = 1$, $\bar{\rho}$ is a well-defined surjective homomorphism and the longitude maps to $\bar{\rho}(l)\bar{\rho}(u) = \lambda$. This implies that λ is realizable. \square

5. THE PONTRYAGIN PRODUCT AND THE SECOND HOMOLOGY GROUP OF A VIRTUAL KNOT

5.1. Pontryagin product. If g_1 and g_2 are commuting elements of a group Π , then their *Pontryagin product* is an element $\langle g_1, g_2 \rangle \in H_2(\Pi)$ defined as follows: Because $[g_1, g_2] = 1$ there is a homomorphism $\phi: \mathbb{Z} \times \mathbb{Z} \rightarrow \Pi$ given by $\phi(1, 0) = g_1$ and $\phi(0, 1) = g_2$. Now $H_2(\mathbb{Z} \times \mathbb{Z}) = \mathbb{Z}$ with generator z described as the image of $1 \otimes 1$ under the cross product isomorphism $H_1(\mathbb{Z}) \otimes H_1(\mathbb{Z}) \rightarrow H_2(\mathbb{Z} \times \mathbb{Z})$ arising in the Künneth formula. Set $\langle g_1, g_2 \rangle = \phi_*(z)$. It can be interpreted geometrically as follows: Since $[g_1, g_2] = 1$ in Π , $[g_1, g_2]$ is nullhomotopic in the Eilenberg-MacLane space $K(\Pi, 1)$. Thus there is a map φ from a torus T^2 to $K(\Pi, 1)$ sending a meridian and a longitude to paths representing g_1 and g_2 . The Pontryagin product $\langle g_1, g_2 \rangle$ is $\varphi_*[T^2]$ in $H_2(K(\Pi, 1))$, where $[T^2]$ is the generator of $H_2(T^2)$. A reference is [1].

5.2. A CW-complex obtained from a realizable Wirtinger presentation.

Let D be a Gauss diagram with group $\Pi = \langle t_1, \dots, t_n \mid t_2^{-1}t_1^{t_{i_1}^{\epsilon_1}}, \dots, t_1^{-1}t_n^{t_{i_n}^{\epsilon_n}} \rangle$. We define a 2-dimensional CW complex whose fundamental group is Π . Consider a torus T^2 as a square identified linearly the top side with the bottom side and the left side with the right side. We divide the square into n rectangles R_1, \dots, R_n by $n - 1$ vertical lines and label the vertical lines t_1, \dots, t_n successively from the left to the right with orientation from the bottom to the top. A CW complex X_D is defined as a quotient of T^2 as follows: For $j = 1, \dots, n$, the top side of each rectangle R_j is oriented from the left to the right and identified linearly with the left vertical line t_{i_j} of the rectangle R_{i_j} in an oriented manner according to the sign

FIGURE 5. Gauss diagram D and space X_D

of ϵ_j . In fact, if D represented a classical knot K in S^3 , X_D is the 2-skeleton of a CW complex of $S^3 \setminus K$. It can be easily checked that Π is the fundamental group of X_D .

5.3. A description on the second homology of a group. As briefly described in Section 2, there is a description on the second homology of a group Π . If Y is a connected CW-complex with $\pi_1(Y) \cong \Pi$ and $\Sigma_2(Y)$ denotes the subgroup of $H_2(Y)$ generated by all singular 2-cycles represented by maps of a 2-sphere into Y , then $H_2(\Pi) = H_2(Y)/\Sigma_2(Y)$. Note $\Sigma_2(Y) = \rho(\pi_2(Y))$, where $\rho: \pi_2(Y) \rightarrow H_2(Y)$ is the Hurewicz homomorphism. Compare [10, 2].

Theorem 15. *The Pontryagin product of the meridian and longitude of a virtual knot generates the second homology of its group.*

Proof. Let D be a representative Gauss diagram of a virtual knot. We first compute the second homology of X_D . Consider the cellular chain complex of X_D

$$0 \rightarrow C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \rightarrow 0,$$

where C_2 , C_1 , and C_0 are the free abelian groups generated by the rectangles R 's, the circles t 's, and the vertex, respectively. The map ∂_1 is a zero map and the map

∂_2 is represented by a matrix $\begin{pmatrix} 1 & 0 & \cdots & -1 \\ -1 & 1 & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$ where $\ker \partial_2$ is generated by

the element $[R_1 + \cdots + R_n]$. Let q be the quotient map from T^2 to X_D . Then $q_*[T^2]$ is the element $[R_1 + \cdots + R_n]$ in $H_2(X_D)$, where $[T^2]$ denotes the generator of $H_2(T^2)$.

Observe that a meridian of the Gauss diagram D is t_1 and its longitude can be exactly read off from the top side of the square of X_D . Since the meridian and the longitude of T^2 map to t_1 and the top side of the square via the quotient map $q: T^2 \rightarrow X_D$, respectively, $q_*[T^2]$ in $H_2(X_D)$ is the Pontryagin product of the meridian and longitude of the diagram D and hence it generates $H_2(X_D)$ and $H_2(\Pi) = H_2(X_D)/\Sigma_2(X_D)$. \square

Corollary 16. *If a virtual knot has a nonzero second homology group, the Pontryagin product of its meridian and longitude is not zero.*

The following is a partial converse of Proposition 6 because the second homology of a Wirtinger presentation of deficiency 1 is zero.

Corollary 17. *If a group Π is the group of a virtual knot with trivial longitude, then $H_2(\Pi) = 0$.*

We conclude this section posing a question: Can any Wirtinger presentation of deficiency 0 with trivial second homology be realized as the group of a virtual knot with trivial longitude?

6. EXAMPLES

6.1. Examples of virtual knot groups with nontrivial second homology.

Edmonds and Livingston [3] showed that the Pontryagin product of the meridian and longitude of a classical knot group represented to a symmetric group S_n is zero, and Johnson and Livingston [8] extended this result to a general group representation. However, there are virtual knots whose second homology is nonzero and hence whose Pontryagin products of meridians and longitudes are not zero. By Theorem 3 and Corollary 16 it suffices to find examples of Wirtinger presentations of deficiency 0 with nontrivial second homology.

6.2. Infinite cyclic second homology. Gordon [6] gave a family of Wirtinger presentations of deficiency 0 whose second homology groups are infinite cyclic \mathbb{Z} . The groups are defined by, for $k \geq 2$,

$$\langle t, x, y \mid t^{-1}x^kt = x^{k+1}, t^{-1}y^kt = y^{k+1}, t^{-1}xyt = xy \rangle$$

Setting $z = tx$ and $w = ty$ gives the Wirtinger presentation

$$\langle t, z, w \mid z = t^{(t^{-1}z)^{-k}}, w = t^{(t^{-1}w)^{-k}}, t = t^{w^{-1}tz^{-1}} \rangle$$

of deficiency 0.

6.3. Order two second homology. Brunner, Mayland, and Simon [2] gave a Wirtinger group of deficiency 0 whose second homology group is cyclic of order 2. The group is defined by

$$\langle a, b \mid b = a^{-1}b^2ab^{-2}a, b = [ba^{-1}, a^{-1}b]^{-1}b[ba^{-1}, a^{-1}b] \rangle$$

which is a Wirtinger presentation of deficiency 0.

I do not know any examples of Wirtinger group presentations of deficiency 0 whose second homology groups are cyclic of finite order other than 2.

6.4. An example of a virtual knot with trivial longitude whose group has deficiency 0.

The group Π with presentation $\langle x, a \mid xa^2 = ax, a^2x = xa \rangle$ considered by Fox in [5], Example 12, has vanishing second homology group by Kervaire's Theorem in [10] because it is the group of some 2-knot. However, this group does not have deficiency 1 (it has therefore deficiency 0) because its Alexander ideal, which is easily computed to be $(3, 1 + t)$, fails to be principal. (For the notion of the Alexander ideal, see [4].) By letting $y = ax$, the presentation is transformed into a Wirtinger presentation $\langle x, y \mid y = xy^{-1}x, x = yxy^{-1} \rangle$. Its longitude is $y^{-1}x^2y^{-1}$. To see that the longitude is the identity in Π , consider the infinite cyclic cover \tilde{X}_Π of the space X_Π . The cover \tilde{X}_Π has a group presentation $\langle x_i, i \in \mathbb{Z} \mid x_i^{-1}x_{i+1}^2, x_j^{-2}x_{j+1} \rangle$, which is isomorphic to $\mathbb{Z}/3\mathbb{Z}$. The longitude $y^{-1}x^2y^{-1}$ lifts to $x_0x_1 = x_0^3 = 1$ which is trivial in $\mathbb{Z}/3\mathbb{Z}$ and hence trivial in Π because $\pi_1(\tilde{X}_\Pi)$ is the commutator subgroup of Π . Therefore, there is a virtual knot with trivial longitude whose group has deficiency 0 and a trivial second homology.

REFERENCES

- [1] K. Brown, *Cohomology of Groups*, GTM 87, Springer-Verlag, New York, 1982.
- [2] A. M. Brunner, E. J. Mayland, Jr., and J. Simon, *Knot groups in S^4 with nontrivial homology*, Pacific J. Math. **103** (1982), no. 2, 315–324.
- [3] A. Edmonds and C. Livingston, *Symmetric representations of knot groups*, Topology Appl. **18** (1984), 281–314.
- [4] R. H. Fox, *Free differential calculus II*, Ann. of Math. **59** (1954), 196–210.
- [5] R. H. Fox, *A quick trip through knot theory*, Topology of 3-manifolds, M. K. Fort, Jr. ed., Prentice-Hall, 1962, 120–167.
- [6] C. Mc. A. Gordon, *Homology of groups of surfaces in the 4-sphere*, Math. Proc. Cambridge Philos. Soc. **89** (1981), no. 1, 113–117.
- [7] M. Goussarov, M. Polyak and O. Viro, *Finite type invariants of classical and virtual knots*, preprint (October 1998 – math.GT/9810073).
- [8] D. J. Johnson and C. Livingston, *Peripherally specified homomorphisms of knot groups*, Trans. Amer. Math. Soc. **311** (1989), no. 1, 135–146.
- [9] L. Kauffman, *Virtual Knot Theory*, preprint (November 1998 – math.GT/9811028).
- [10] M. A. Kervaire, *On higher dimensional knots*, Differential and Combinatorial Topology, A Symposium in Honor of Marston Morse, S. S. Cairns ed., Princeton University Press, Princeton, New Jersey, 1965, 105–119.

DEPARTMENT OF MATHEMATICS, INDIANA UNIVERSITY, BLOOMINGTON, INDIANA 47405

E-mail address: `sekim@indiana.edu`